## IB MATHEMATICS SL

## TOPIC: Vectors

## VECTOR ALGEBRA

1. The vectors $\boldsymbol{u}, \boldsymbol{v}$ are given by $\boldsymbol{u}=3 \boldsymbol{i}+5 \boldsymbol{j}, \boldsymbol{v}=\boldsymbol{i}-2 \boldsymbol{j}$.

Find scalars $a, b$ such that $a(\boldsymbol{u}+\boldsymbol{v})=8 \boldsymbol{i}+(b-2) \boldsymbol{j}$.

Working:

Answer:
2. The following diagram shows the point $O$ with coordinates $(0,0)$, the point A with position vector $\boldsymbol{a}=12 \boldsymbol{i}$ $+5 \boldsymbol{j}$, and the point B with position vector $\boldsymbol{b}=6 \boldsymbol{i}+8 \boldsymbol{j}$. The angle between ( OA ) and $(\mathrm{OB})$ is $\theta$.

## Diagram not to scale



Find
(i) $|\boldsymbol{a}|$;
(ii) a unit vector in the direction of $\boldsymbol{b}$;
(iii) the exact value of $\cos \theta$ in the form $\frac{p}{q}$, where, $p, q \in \mathbb{Z}$.
3. Calculate the acute angle between the lines with equations
$\boldsymbol{r}=\binom{4}{-1}+s\binom{4}{3}$ and $\quad \boldsymbol{r}=\binom{2}{4}+t\binom{1}{-1}$

## Working:

Answer:
4. Find a vector equation of the line passing through $(-1,4)$ and (3, -1 ). Give your answer in the form $\boldsymbol{r}=\boldsymbol{p}$ $+t d$, where $t \in \mathbb{R}$.
$\square$
Answer:
$\qquad$
5. A line passes through the point $(4,-1)$ and its direction is perpendicular to the vector $\binom{2}{3}$. Find the equation of the line in the form $a x+b y=p$, where $a, b$ and $p$ are integers to be determined.


Answer:
6. Two lines $L_{1}$ and $L_{2}$ have these vector equations.
$L_{1}: \boldsymbol{r}=2 \boldsymbol{i}+3 \boldsymbol{j}+t(\boldsymbol{i}-3 \boldsymbol{j})$
$L_{2}: \boldsymbol{r}=\boldsymbol{i}+2 \boldsymbol{j}+s(\boldsymbol{i}-\boldsymbol{j})$
The angle between $L_{1}$ and $L_{2}$ is $\theta$. Find the cosine of the angle $\theta$.
$\square$
Answer:
7. The points A and B have the position vectors $\binom{2}{-2}$ and $\binom{-3}{-1}$ respectively.
(a) (i) Find the vector $\overrightarrow{\mathrm{AB}}$.
(ii) Find $|\overrightarrow{\mathrm{AB}}|$.

The point D has position vector $\binom{d}{23}$
(b) Find the vector $\overrightarrow{\mathrm{AD}}$ in terms of $d$.

The angle BÂD is $90^{\circ}$.
(c) (i) Show that $d=7$.
(ii) Write down the position vector of the point D .

The quadrilateral ABCD is a rectangle.
(d) Find the position vector of the point C .
(e) Find the area of the rectangle ABCD .
8. The circle shown has centre $O$ and radius $6 . \overrightarrow{O A}$ is the vector $\binom{6}{0}, \overrightarrow{O B}$ is the vector $\binom{-6}{0}$ and $\overrightarrow{O C}$ is the vector $\binom{5}{\sqrt{11}}$.

(a) Verify that $A, B$ and $C$ lie on the circle.
(b) Find the vector $\overrightarrow{A C}$.
(c) Using an appropriate scalar product, or otherwise, find the cosine of angle $O \hat{A} C$.
(d) Find the area of triangle $A B C$, giving your answer in the form $a \sqrt{11}$, where $a \in \mathbb{N}$.
9. Three of the coordinates of the parallelogram STUV are $S(-2,-2), T(7,7), U(5,15)$.
(a) Find the vector $\overrightarrow{\mathrm{ST}}$ and hence the coordinates of V .
(b) Find a vector equation of the line (UV) in the form $\boldsymbol{r}=\boldsymbol{p}+\lambda \mathbf{d}$ where $\lambda \in \mathbb{R}$.
(c) Show that the point $E$ with position vector $\binom{1}{11}$ is on the line (UV), and find the value of $\lambda$ for this point.

The point W has position vector $\binom{a}{17}, a \in \mathbb{R}$.
(d) (i) If $|\overrightarrow{\mathrm{EW}}|=2 \sqrt{13}$, show that one value of $a$ is -3 and find the other possible value of $a$.
(ii) For $a=-3$, calculate the angle between $\overrightarrow{\mathrm{EW}}$ and $\overrightarrow{\mathrm{ET}}$.

## VECTOR APPLICATION

10. In this question the vector $\binom{1}{0} \mathrm{~km}$ represents a displacement due east, and the vector $\binom{0}{1} \mathrm{~km}$ represents a displacement due north.

The diagram shows the path of the oil-tanker Aristides relative to the port of Orto, which is situated at the point ( 0,0 ).


The position of the Aristides is given by the vector equation

$$
\binom{x}{y}=\binom{0}{28}+t\binom{6}{-8}
$$

at a time t hours after 12:00.
(a) Find the position of the Aristides at 13:00.
(b) Find
(i) the velocity vector;
(ii) the speed of the Aristides.
(c) Find a cartesian equation for the path of the Aristides in the form

$$
\begin{equation*}
a x+b y=g \tag{4}
\end{equation*}
$$

Another ship, the cargo-vessel Boadicea, is stationary, with position vector $\binom{18}{4} \mathrm{~km}$.
(d) Show that the two ships will collide, and find the time of collision.

To avoid collision, the Boadicea starts to move at 13:00 with velocity vector $\binom{5}{12} \mathrm{~km} \mathrm{~h}^{-1}$.
(e) Show that the position of the Boadicea for $t \geq 1$ is given by

$$
\binom{x}{y}=\binom{13}{-8}+t\binom{5}{12}
$$

(f) Find how far apart the two ships are at 15:00.
11. In this question, the vector $\binom{1}{0} \mathrm{~km}$ represents a displacement due east, and the vector $\binom{0}{1} \mathrm{~km} \mathrm{a}$ displacement due north.

Two crews of workers are laying an underground cable in a north-south direction across a desert. At 06:00 each crew sets out from their base camp which is situated at the origin $(0,0)$. One crew is in a Toyundai vehicle and the other in a Chryssault vehicle.

The Toyundai has velocity vector $\binom{18}{24} \mathrm{~km} \mathrm{~h}^{-1}$, and the Chryssault has velocity vector $\binom{36}{-16} \mathrm{~km} \mathrm{~h}^{-1}$.
(a) Find the speed of each vehicle.
(b) (i) Find the position vectors of each vehicle at 06:30.
(ii) Hence, or otherwise, find the distance between the vehicles at 06:30.
(c) At this time (06:30) the Chryssault stops and its crew begin their day's work, laying cable in a northerly direction. The Toyundai continues travelling in the same direction at the same speed until it is exactly north of the Chryssault. The Toyundai crew then begin their day's work, laying cable in a southerly direction. At what time does the Toyundai crew begin laying cable?
(d) Each crew lays an average of 800 m of cable in an hour. If they work non-stop until their lunch break at 11:30, what is the distance between them at this time?
(e) How long would the Toyundai take to return to base camp from its lunch-time position, assuming it travelled in a straight line and with the same average speed as on the morning journey? (Give your answer to the nearest minute.)
12. The diagram below shows the positions of towns $O, A, B$ and $X$.

## Diagram not to scale



Town A is 240 km East and 70 km North of O.
Town B is 480 km East and 250 km North of O.
Town X is 339 km East and 238 km North of O.

An airplane flies at a constant speed of $300 \mathrm{~km} \mathrm{~h}^{-1}$ from O towards A .
(a) (i) Show that a unit vector in the direction of $\overrightarrow{\mathrm{OA}}$ is $\binom{0.96}{0.28}$.
(ii) Write down the velocity vector for the airplane in the form $\binom{v_{1}}{v_{2}}$.
(iii) How long does it take for the airplane to reach A?

At A the airplane changes direction so it now flies towards B . The angle between the original direction and the new direction is $\theta$ as shown in the following diagram. This diagram also shows the point Y , between A and B , where the airplane comes closest to X .

## Diagram not to scale


(b) Use the scalar product of two vectors to find the value of $\theta$ in degrees.
(c) (i) Write down the vector $\overrightarrow{\mathrm{AX}}$.
(ii) Show that the vector $\boldsymbol{n}=\binom{-3}{4}$ is perpendicular to $\overrightarrow{\mathrm{AB}}$.
(iii) By finding the projection of $\overrightarrow{\mathrm{AX}}$ in the direction of $\boldsymbol{n}$, calculate the distance XY.
(d) How far is the airplane from A when it reaches Y ?
13. In this question, a unit vector represents a displacement of 1 metre.

A miniature car moves in a straight line, starting at the point $(2,0)$. After $t$ seconds, its position, $(x, y)$, is given by the vector equation

$$
\binom{x}{y}=\binom{2}{0}+t\binom{0.7}{1}
$$

(a) How far from the point $(0,0)$ is the car after 2 seconds?
(b) Find the speed of the car.
(c) Obtain the equation of the car's path in the form $a x+b y=c$.

Another miniature vehicle, a motorcycle, starts at the point ( 0,2 ), and travels in a straight line with constant speed. The equation of its path is

$$
y=0.6 x+2, \quad x \geq 0
$$

Eventually, the two miniature vehicles collide.
(d) Find the coordinates of the collision point.
(e) If the motorcycle left point $(0,2)$ at the same moment the car left point $(2,0)$, find the speed of the motorcycle.

## 3D VECTORS

14. Consider the vectors $\boldsymbol{u}=2 \boldsymbol{i}+3 \boldsymbol{j}-\boldsymbol{k}$ and $\boldsymbol{v}=4 \boldsymbol{i}+\boldsymbol{j}-\boldsymbol{k} \boldsymbol{k}$.
(a) Given that $\boldsymbol{u}$ is perpendicular to $\boldsymbol{v}$ find the value of $p$.
(b) Given that $q|\mathbf{u}|=14$, find the value of $q$.
$\qquad$
$\qquad$
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15. The diagram shows a cube, OABCDEFG where the length of each edge is 5 cm . Express the following vectors in terms of $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$.

(a) $\overrightarrow{\mathrm{OG}}$;
(b) $\overrightarrow{\mathrm{BD}}$;
(c) $\overrightarrow{\mathrm{EB}}$
$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$
$\qquad$
16. Consider the points $A(1,5,4), B(3,1,2)$ and $D(3, k, 2)$, with $(A D)$ perpendicular to $(A B)$.
(a) Find
(i) $\overrightarrow{\mathrm{AB}}$;
(ii) $\overrightarrow{\mathrm{AD}}$, giving your answer in terms of $k$.
(b) Show that $k=7$.

The point $C$ is such that $\overrightarrow{B C}=\frac{1}{2} \overrightarrow{\mathrm{AD}}$.
(c) Find the position vector of C.
(d) Find $\cos \mathrm{A} \hat{B} C$.
17. In this question, distance is in kilometers, time is in hours.

A balloon is moving at a constant height with a speed of $18 \mathrm{~km} \mathrm{~h}^{-1}$, in the
direction of the vector $\left(\begin{array}{l}3 \\ 4 \\ 0\end{array}\right)$

At time $t=0$, the balloon is at point B with coordinates $(0,0,5)$.
(a) Show that the position vector $\boldsymbol{b}$ of the balloon at time $t$ is given by

$$
\boldsymbol{b}=\left(\begin{array}{l}
x \\
y \\
z
\end{array}\right)=\left(\begin{array}{l}
0 \\
0 \\
5
\end{array}\right)+t\left(\begin{array}{c}
10.8 \\
14.4 \\
0
\end{array}\right) .
$$

At time $t=0$, a helicopter goes to deliver a message to the balloon. The position vector $\boldsymbol{h}$ of the helicopter at time $t$ is given by

$$
\boldsymbol{h}=\left(\begin{array}{l}
x \\
y \\
z
\end{array}\right)=\left(\begin{array}{c}
49 \\
32 \\
0
\end{array}\right)+t\left(\begin{array}{c}
-48 \\
-24 \\
6
\end{array}\right)
$$

(b) (i) Write down the coordinates of the starting position of the helicopter.
(ii) Find the speed of the helicopter.
(c) The helicopter reaches the balloon at point R .
(i) Find the time the helicopter takes to reach the balloon.
(ii) Find the coordinates of R.
18. The position vector of point A is $2 \boldsymbol{i}+3 \boldsymbol{j}+\boldsymbol{k}$ and the position vector of point B is $4 \boldsymbol{i}-5 \boldsymbol{j}+21 \boldsymbol{k}$.
(a) (i) Show that $\overrightarrow{\mathrm{AB}}=2 \boldsymbol{i}-8 \boldsymbol{j}+20 \boldsymbol{k}$.
(ii) Find the unit vector $\boldsymbol{u}$ in the direction of $\overrightarrow{\mathrm{AB}}$.
(iii) Show that $\boldsymbol{u}$ is perpendicular to $\overrightarrow{\mathrm{OA}}$.

Let $S$ be the midpoint of $[\mathrm{AB}]$. The line $L_{1}$ passes through S and is parallel to $\overrightarrow{\mathrm{OA}}$.
(b) (i) Find the position vector of $S$.
(ii) Write down the equation of $L_{1}$.

The line $L_{2}$ has equation $\boldsymbol{r}=(5 \boldsymbol{i}+10 \boldsymbol{j}+10 \boldsymbol{k})+s(-2 \boldsymbol{i}+5 \boldsymbol{j}-3 \boldsymbol{k})$.
(c) Explain why $L_{1}$ and $L_{2}$ are not parallel.
(d) The lines $L_{1}$ and $L_{2}$ intersect at the point P . Find the position vector of P .

